Math 279 Lecture 11 Notes

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1 Gaussian Inequalities and Markov Techniques for Lifts of Brownian Motion

1.1 Gaussian-type inequalities

For many stochastic processes of interest, we either can use the Markov property or take advantage of the Gaussian distribution of the realizations. In the former case, many martingales become available, and in the latter case, many Gaussian-type inequalities can be used.

For example, if $x : [0, T] \to \mathbb{R}^d$ is a Gaussian process that is centered (i.e. $\mathbb{E}[x(t)] = 0$ for all t), the process is determined by its correlation, $\mathbb{E}[x(t) \otimes x(s)] = R(s, t)$. For simplicity, let us assume that $x = (x_1, \dots, x_d)$ with x_i, x_j independent for $i \neq j$. Then $R(s, t)$ is diagonal.

Example 1.1. If X_i have the same law for $i = 1, \dots, d$, then $R(s, t) = C(s, t)I$, where C is scalar-valued, and I is the identity matrix. Also,

$$\mathbb{E}[|x_i(t) - x_i(s)|^2] = C(t, t) + C(s, s) - 2C(s, t),$$

and if

$$\mathbb{E}[|x_i(t) - x_i(s)|^2] \le c_0|t - s|^{2\alpha},$$

then we can use Kolmogorov's continuity theorem to assert that $x \in \mathcal{C}^\beta$ for every $\beta < \alpha$. Indeed, this estimate would imply that

$$\begin{aligned}(\mathbb{E}[|x_i(t) - x_i(s)|^{2n}])^{1/2n} &\le a_n(\mathbb{E}[|x_i(t) - x_i(s)|^2])^{1/2} \\ &\le \sqrt{c_0}a_n|t - s|^\alpha,\end{aligned}$$

and we can use Kolmogorov's continuity theorem to obtain control on $[x_i]_{\alpha-1/(2n)-\varepsilon}$; this holds for any n. To see this, observe that if X is normal with mean 0 and $\mathbb{E}[X^2] = A$, then $\mathbb{E}[e^{tX}] = e^{\frac{t^2}{2}A}$, so that

$$\mathbb{E}[X^{2n}] = \frac{(2n)!}{n!2^n}(\mathbb{E}[X^2])^n.$$

The moral is that in the Gaussian case, we can bound higher moments in terms of the second moment. The good news is that something similar is also true for martingales.

Rough path theory can be carried out for any Gaussian process, provided that $\mathbb{E}[|x_i(t)-x_i(s)|^2] \leq c_0|t-s|^{2\alpha}$ for some $\alpha > 0$. For example, we can consider a fractional Brownian motion that is specified by the requirement that $\mathbb{E}[|x_i(t)-x_i(s)|^2] = |t-s|^{2H}$, where H is known as the **Hurst index**.

1.2 Brownian motion as a Markov process

How about the Brownian motion as a Markov process? Let $B = (B_1, \dots, B_d)$, where the B_is are independent standard Brownian motion. As we discussed last time, we can come up with a candidate for

$$\int_s^t B_i \, dB_j = \lim_{n\to\infty} \sum_{k:t_k^n \in [s,t]} B_i(t_k^n) B_j(t_k^n, t_{k+1}^n), \quad \text{where} \quad D_n = \{t_k^n = k/2^n : k \in \mathbb{Z}\}.$$

This is in $L^2(\mathbb{P})$, where $\mathbb{P}$ is **Wiener measure**, a probability measure on $C([0,T];\mathbb{R}^d)$. We had another candidate that we will denote

$$\int_s^t B_i \circ dB_j := \lim_{n\to\infty} \sum_{k:t_k^n \in [s,t]} \frac{B_i(t_k^n) + B_i(t_{k+1}^n)}{2} B_k(t_k^n, t_{k+1}^n).$$

For diagonal terms, we have explicit formulae, namely

$$\int_s^t B_i \, dB_i = \frac{B_i(t)^2 - B_i(s)^2}{2} - \frac{t-s}{2}, \qquad \int_s^t B_i \circ dB_i = \frac{B_i(t)^2 - B_i(s)^2}{2}.$$

Though when $i \neq j$, we have $\int_s^t B_i \, dB_j = \int_s^t B_i \circ dB_j$ because

$$\int_s^t B_i \circ dB_j - \int_s^t B_i \, dB_j = \lim_{n\to\infty} \frac{1}{2} \sum_{t_k^n \in [s,t]} B_i(t_k^n, t_{k+1}^n) B_j(t_k^n, t_{k+1}^n),$$

and

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{t_k^n \in [s,t]} B_i(t_k^n, t_{k+1}^n) B_j(t_k^n, t_{k+1}^n)\right)^2\right] &= \sum_{t_k^n \in [s,t]} \mathbb{E}[B_i(t_k^n, t_{k+1}^n)^2] \, \mathbb{E}[B_j(t_k^n, t_{k+1}^n)^2] \\ &\approx 2^n (t-s) 2^{-n} 2^{-n} \\ &\to 0.\end{aligned}$$

In summary,

$$\mathbb{B}^{\text{Itô}}(s,t) = \mathbb{B}^{\text{Strat}}(s,t) - \frac{1}{2}(t-s)I,$$

where I is the identity matrix.

However, we need to show that $(B, \mathbb{B}^{\text{Itô}}) \in \mathscr{R}^\alpha$ for any $\alpha < 1/2$. We have done with the B part. We get our estimate for $\mathbb{B}^{\text{Itô}}$ using the fact that $M_{i,j}(t) = \int_0^t B_i \, dB_j$ is a martingale. We write $\mathcal{F}_t$ for the σ-algebra generated by $(B(\theta) : \theta \in [0,t])$. Then $M(t)$ is a **martingale** if $\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$, or $\mathbb{E}[M(t) - M(s) \mid \mathcal{F}_s] = 0$.

For example, $B(t)$ itself is a martingale, and observe that $\mathbb{E}[\int_s^t B_i \, dB_j \mid \mathcal{F}_s] = 0$. Indeed,

$$\begin{aligned}
\mathbb{E}\left[\sum_{k/2^n \in [s,t]} B_i(t_k^n) B_k(t_k^n, t_{k+1}^n) \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\sum_{k/2^n \in [s,t]} B_i(s) B_j(t_k^n, t_{k+1}^n) \mid \mathcal{F}_s\right] \\
&\approx \mathbb{E}[B_i(s) B_j(s,t)] \\
&= 0.
\end{aligned}$$

First, we can show that

$$\mathbb{E}[M_{i,j}(t)^2] = \mathbb{E}\left[\int_0^t B_i(s)^2 \, ds\right],$$

which yields

$$\begin{aligned}
\mathbb{E}[(B_{i,j}^{\text{Itô}})^2] &= \mathbb{E}\left[\left(\int_s^t (B_i(\theta) - B_i(s)) \, dB_j(\theta)\right)^2\right] \\
&= \mathbb{E}\left[\int_s^t (B_i(\theta) - B_i(s))^2 \, d\theta\right] \\
&= \int_s^t (\theta - s) \, d\theta \\
&= \frac{(t-s)^2}{2}.
\end{aligned}$$

Here, if we write $A_{i,j}(t) = \int_0^t B_i(\theta)^2 \, d\theta$, then $M_{i,j}(t)^2 - A_{i,j}(t)$ is again a martingale.[1] We have the following fundamental inequality in this context that is due to Burkholder-Davis-Gundy (Doob's inequality):

Lemma 1.1. *If M and $M^2 - \langle M \rangle = M^2 - [M] = M^2 - A$ are martingales with $M(0) = 0$, define $M^*(t) = \sup_{s \in [0,t]} |M(s)|$. Then*

$$\mathbb{E}[M^*(t)^q] \leq c_q \, \mathbb{E}[A^{q/2}].$$

[1] This is not a coincidence. For any such martingale, if we square it, there is a monotone function we can subtract to get another martingale.

Now, for our example,

$$\begin{aligned}
\mathbb{E}[|\mathbb{B}^{\text{Itô}}(s,t)|^q] &\le c\,\mathbb{E}\left[\left|\int_s^t B_i(s,\theta)^2\,d\theta\right|^{q/2}\right] \\
&\le c\,\mathbb{E}\left[\left(\sup_{\theta\in[s,t]} |B_i(s,\theta)|\right)^q\right] |t-s|^{q/2} \\
&\le c'|t-s|^{\alpha q}|t-s|^{q/2}.
\end{aligned}$$

So

$$(\mathbb{E}[|\mathbb{B}^{\text{Itô}}(s,t)|^q])^{1/q} \le c|t-s|^{\alpha+1/2}.$$